

# Tight Linear Convergence Rate Bounds for Douglas-Rachford Splitting and ADMM

Pontus Giselsson\*

**Abstract**—Douglas-Rachford splitting and the alternating direction method of multipliers (ADMM) can be used to solve convex optimization problems that consist of a sum of two functions. Convergence rate estimates for these algorithms have received much attention lately. In particular, linear convergence rates have been shown by several authors under various assumptions. One such set of assumptions is strong convexity and smoothness of one of the functions in the minimization problem. The authors recently provided a linear convergence rate bound for such problems. In this paper, we show that this rate bound is tight for many algorithm parameter choices.

## I. INTRODUCTION

Douglas-Rachford splitting is an optimization algorithm that can solve general convex composite optimization problems. The algorithm has its roots in the 1950's [5], [17]. In the late 1970's, it was shown [14] how to use the algorithm to solve monotone operator inclusion problems and convex composite optimization problems. The alternating direction method of multipliers (ADMM) can also solve composite optimization problems. It was first presented in [11], [7]. Soon thereafter, it was shown [6] that ADMM is equivalent to Douglas-Rachford splitting applied to the dual problem.

General sublinear convergence rate estimates for these methods have just recently been presented in the literature, see [12], [3], [1]. Under various assumptions, also linear convergence rates can be established. In the paper by Lions and Mercier [14], a linear convergence rate was provided for Douglas-Rachford splitting under (the equivalence of) strong convexity and smoothness assumptions. Until recently, further linear convergence rate results have been scarce. The last couple of years, however, several linear convergence rate results for both Douglas-Rachford splitting and ADMM have been presented. These include [4], [2], in which linear convergence rates for ADMM are presented under various assumptions. In [13], linear convergence rates are established for multiple splitting ADMM. In [16], it is shown that for a specific class of problems, the Douglas-Rachford algorithm can be interpreted as a gradient method of a function named the Douglas-Rachford envelope. By showing strong convexity and smoothness properties of the Douglas-Rachford envelope under similar assumptions on the underlying problem, a linear convergence rate is established based on gradient algorithm theory. Very recently [15] appeared and showed linear convergence of ADMM under smoothness and strong convexity assumptions using the integral quadratic constraints (IQC) framework. The rate is obtained by solving

a series of a small semi-definite programs. Common for all these linear convergence rate bounds are that they are not tight for the class of problems under consideration, see [10, Section IV.B].

In [18], linear convergence of ADMM is established under more general assumptions than the above. However, the assumptions are more difficult to verify for a given problem. Tightness is verified for a 2-dimensional example in the Euclidean case. In [8], linear convergence for ADMM on strongly convex quadratic optimization problem with inequality constraints is established. This rate improves on the rates presented in [14], [4], [2], [13], [16], [15]. In [9], the authors generalize, using a completely different machinery, the results in [8] and in [10] the results are further generalized. More specifically, [10] generalizes the results in [8] in the following three ways; (i) a wider class of problems is considered, (ii) rates for both Douglas-Rachford splitting and ADMM are provided, and (iii) the results in [10] hold for general real Hilbert spaces as opposed to the Euclidean space only in [8]. For the restricted class of problems considered in [8], the convergence rate bounds in [10] and [8] coincide.

The contribution of this paper is that we show tightness of the convergence rate bounds presented in [10] for the class of problems under consideration and for many algorithm parameters. This is done by formulating examples, both for Douglas-Rachford splitting and ADMM, for which the linear convergence rate bounds are satisfied with equality. Similar lower convergence rate bounds have been presented in [15]. The bounds in this paper cover wider classes of problems and are less conservative.

## II. NOTATION

We denote by  $\mathbf{R}$  the set of real numbers,  $\mathbf{R}^n$  the set of real column-vectors of length  $n$ . Further  $\bar{\mathbf{R}} := \mathbf{R} \cup \{\infty\}$  denotes the extended real line. Throughout this paper  $\mathcal{H}$  denotes a real separable Hilbert space. Its inner product is denoted by  $\langle \cdot, \cdot \rangle$ , the induced norm by  $\| \cdot \|$ , and the identity operator by  $\text{Id}$ . The indicator function for a set  $\mathcal{X}$  is denoted by  $\iota_{\mathcal{X}}$ . Finally, the class of closed, proper, and convex functions  $f : \mathcal{H} \rightarrow \bar{\mathbf{R}}$  is denoted by  $\Gamma_0(\mathcal{H})$ .

## III. PRELIMINARIES

In this section we present, well known concepts, results, operators, and algorithms that will be extensively used in the paper.

*Definition 1 (Orthonormal basis):* An orthonormal basis  $\{\phi_i\}_{i=1}^K$  for a (separable) Hilbert space  $\mathcal{H}$  is an orthogonal

\* Department of Automatic Control, Lund University. Email: pontusg@control.lth.se

basis, i.e.  $\langle \phi_i, \phi_j \rangle = 0$  if  $i \neq j$ , where each basis vector has unit length, i.e.  $\|\phi_i\| = 1$ .

Hereon,  $\phi_i$  will denote elements of an orthonormal basis.

*Remark 1:* The number of elements in the basis (the cardinality)  $K$  is equal to the dimension of the corresponding Hilbert space, which might be  $\infty$ . Also, by definition of a basis, each element  $x \in \mathcal{H}$  can be (uniquely) decomposed as  $x = \sum_{i=1}^K \langle x, \phi_i \rangle \phi_i$ , see [20, Proposition 3.3.10].

The reason why we consider separable Hilbert spaces is the following proposition which can be found, e.g., in [20, Proposition 3.3.12].

*Proposition 1:* A Hilbert space is separable if and only if it has an orthonormal basis.

We will also make extensive use of the following two propositions that are proven, e.g., in [20, Proposition 3.3.10] and [20, Proposition 3.3.14] respectively.

*Proposition 2 (Parseval's identity):* In separable Hilbert spaces  $\mathcal{H}$ , the squared norm of each element  $x \in \mathcal{H}$  satisfies

$$\|x\|^2 = \sum_{i=1}^K |\langle x, \phi_i \rangle|^2.$$

*Proposition 3 (Riesz-Fischer):* In separable Hilbert spaces  $\mathcal{H}$ , the sequence  $\sum_{i=1}^{\infty} a_i \phi_i$  converges if and only if  $\sum_{i=1}^{\infty} a_i^2 < \infty$ . Then

$$\left\| \sum_{i=1}^K a_i \phi_i \right\|^2 = \sum_{i=1}^K a_i^2.$$

*Definition 2 (Strong convexity):* A function  $f \in \Gamma_0(\mathcal{H})$  is  $\sigma$ -strongly convex if

$$f(x) \geq f(y) + \langle u, x - y \rangle + \frac{\sigma}{2} \|x - y\|^2$$

holds for all  $x, y \in \mathcal{H}$  and all  $u \in \partial f(y)$ .

*Definition 3 (Smoothness):* A function  $f \in \Gamma_0(\mathcal{H})$  is  $\beta$ -smooth if it is differentiable and

$$f(x) \leq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\beta}{2} \|x - y\|^2 \quad (1)$$

holds for all  $x, y \in \mathcal{H}$ .

*Definition 4 (Proximal operators):* The proximal operator of a function  $f \in \Gamma_0(\mathcal{H})$  is defined as

$$\text{prox}_{\gamma f}(y) := \argmin_x \left\{ f(x) + \frac{1}{2\gamma} \|x - y\|^2 \right\}.$$

*Definition 5 (Reflected proximal operators):* The reflected proximal operator to  $f \in \Gamma_0(\mathcal{H})$  is defined as

$$R_{\gamma f} := 2\text{prox}_{\gamma f} - \text{Id}.$$

*Definition 6 (Fixed-point):* A point  $y \in \mathcal{H}$  is a fixed-point to the (single-valued) operator  $A : \mathcal{H} \rightarrow \mathcal{H}$  if

$$y = Ay.$$

The set of fixed-points to  $A$  is denoted by  $\text{fix} A$ .

*Algorithm 1 (Generalized Douglas-Rachford splitting):* The generalized Douglas-Rachford splitting algorithm is given by the iteration

$$z^{k+1} = (1 - \alpha)\text{Id} + \alpha R_{\gamma g} R_{\gamma f} z^k \quad (2)$$

where  $\alpha \in (0, 1)$  and  $\gamma > 0$  are algorithm parameters.

*Remark 2:* In the general case,  $\alpha$  is restricted to the interval  $(0, 1)$ . Under the assumptions used in this paper, a larger  $\alpha$  can be used as well, see [10].

#### IV. LINEAR CONVERGENCE RATES

In this section, we state the linear convergence rate results for Douglas-Rachford and ADMM in [10]. The paper [10] considers optimization problems of the form

$$\text{minimize} \quad f(x) + g(\mathcal{A}x) \quad (3)$$

where  $x \in \mathcal{H}$ , and  $f$ ,  $g$ , and  $\mathcal{A}$  satisfy the following assumptions:

*Assumption 1:*

- (i) The function  $f \in \Gamma_0(\mathcal{H})$  is  $\sigma$ -strongly convex and  $\beta$ -smooth.
- (ii) The function  $g \in \Gamma_0(\mathcal{K})$ .
- (iii)  $\mathcal{A} : \mathcal{H} \rightarrow \mathcal{K}$  is a surjective bounded linear operator.

Under the additional assumption that  $\mathcal{A} = \text{Id}$  (which implies that  $\mathcal{K} = \mathcal{H}$ ), Douglas-Rachford splitting can be applied to solve (3). It enjoys a linear convergence rate, as shown in [10, Theorem 1]. This result is restated here for convenience.

*Theorem 1:* Suppose that Assumption 1 holds and that  $\mathcal{A} = \text{Id}$ . Then the generalized Douglas Rachford algorithm (Algorithm 1) converges linearly towards a fixed-point  $\bar{z} \in \text{fix}(R_{\gamma f} R_{\gamma g})$  with at least rate  $|1 - \alpha| + \alpha \max\left(\frac{\gamma\beta-1}{\gamma\beta+1}, \frac{1-\gamma\sigma}{1+\gamma\sigma}\right)$ , i.e.

$$\|z^{k+1} - \bar{z}\| \leq \left( |1 - \alpha| + \alpha \max\left(\frac{\gamma\beta-1}{\gamma\beta+1}, \frac{1-\gamma\sigma}{1+\gamma\sigma}\right) \right)^k \|z^0 - \bar{z}\|$$

for any  $\gamma > 0$  and  $\alpha \in (0, \frac{2}{1 + \max(\frac{1-\gamma\sigma}{1+\gamma\sigma}, \frac{\gamma\beta-1}{1+\gamma\beta})})$ .

*Remark 3:* The bound on the rate in Theorem 1 can be optimized with respect to the algorithm parameters  $\alpha$  and  $\gamma$ . The optimal parameters are given by  $\alpha = 1$  and  $\gamma = \frac{1}{\sqrt{\beta\sigma}}$  which yields rate bound factor  $\frac{\sqrt{\beta/\sigma}-1}{\sqrt{\beta/\sigma}+1}$ , see [10, Proposition 16].

In the case where  $\mathcal{A} \neq \text{Id}$ , problem (3) can be solved by applying Douglas-Rachford splitting on the dual problem:

$$\text{minimize} \quad d(\mu) + g^*(\mu) \quad (4)$$

where  $g^* \in \Gamma_0(\mathcal{K})$ , and  $d \in \Gamma_0(\mathcal{K})$  is defined as

$$d := f^* \circ (-\mathcal{A}^*).$$

If the dual problem (4) satisfies Assumption 1 (with  $d$  instead of  $f$  and  $g^*$  instead of  $g$ ), Douglas-Rachford splitting can be applied to solve (4), and Theorem 1 would guarantee a linear convergence rate. Since  $g \in \Gamma_0(\mathcal{K})$ , we have  $g^* \in \Gamma_0(\mathcal{K})$  [19, Theorem 12.2], and we have  $\mathcal{A}$  in Assumption 1(iii)

equal to Id in (4). The remaining assumption needed to apply Theorem 1 is that  $d \in \Gamma_0(\mathcal{K})$  is strongly convex and smooth. Indeed, this is the case as shown in [10, Proposition 18]. This result is restated here for convenience of the reader.

**Proposition 4:** Suppose that Assumption 1 holds. Then  $d \in \Gamma_0(\mathcal{K})$  is  $\frac{\|\mathcal{A}^*\|^2}{\sigma}$ -smooth and  $\frac{\theta^2}{\beta}$ -strongly convex, where  $\theta > 0$  always exists and satisfies  $\|\mathcal{A}^*\mu\| \geq \theta\|\mu\|$  for all  $\mu \in \mathcal{K}$ .

It is well known [6] that Douglas-Rachford splitting applied to the dual problem (4) is equivalent to ADMM applied to the primal problem (3). Therefore, the linear convergence rate obtained by applying Douglas-Rachford splitting to the dual problem (4) directly translates to a linear convergence rate for ADMM. This linear convergence rate bound is stated in [10, Corollary 2], and restated here for convenience.

**Proposition 5:** Suppose that Assumption 1 holds and that generalized Douglas-Rachford is applied to solve the dual problem (4). Then the Douglas-Rachford splitting algorithm converges linearly towards a fixed-point  $\bar{z} \in \text{fix}(R_{\gamma d} R_{\gamma g^*})$  with at least rate  $|1 - \alpha| + \alpha \max\left(\frac{\gamma\hat{\beta}-1}{\gamma\hat{\beta}+1}, \frac{1-\gamma\hat{\sigma}}{1+\gamma\hat{\sigma}}\right)$ , i.e.

$$\|z^{k+1} - \bar{z}\| \leq \left(|1 - \alpha| + \alpha \max\left(\frac{\gamma\hat{\beta}-1}{\gamma\hat{\beta}+1}, \frac{1-\gamma\hat{\sigma}}{1+\gamma\hat{\sigma}}\right)\right)^k \|z^0 - \bar{z}\|$$

for any  $\gamma > 0$  and  $\alpha \in (0, \frac{2}{1 + \max(\frac{1-\gamma\hat{\sigma}}{1+\gamma\hat{\sigma}}, \frac{\gamma\hat{\beta}-1}{1+\gamma\hat{\beta}})})$ , where

$$\hat{\beta} = \frac{\|\mathcal{A}^*\|^2}{\sigma} \text{ and } \hat{\sigma} = \frac{\theta^2}{\beta}.$$

**Remark 4:** The parameters that optimize the convergence rate bound are  $\alpha = 1$  and  $\gamma = \frac{1}{\sqrt{\hat{\beta}\hat{\sigma}}} = \frac{\sqrt{\beta\sigma}}{\sqrt{\|\mathcal{A}^*\|^2\theta^2}}$  and the linear convergence rate bound factor is  $\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}$ , where  $\kappa = \frac{\hat{\beta}}{\hat{\sigma}} = \frac{\|\mathcal{A}^*\|^2\beta}{\theta^2\sigma}$ , see [10, Corollary 2].

## V. TIGHTNESS OF RATE BOUNDS

In this section, we will state examples that show tightness of the linear convergence rate bounds in Theorem 1 and Proposition 5 for many choices of algorithm parameters.

### A. Primal Douglas-Rachford splitting

To establish that the convergence rate bound provided in [10, Theorem 1] and restated in Theorem 1 is tight, we consider a problem of the form (3) with

$$f(x) = \sum_{i=1}^K \frac{\lambda_i}{2} \langle x, \phi_i \rangle^2, \quad (5)$$

$$g(x) = 0, \quad (6)$$

$$\mathcal{A} = \text{Id}. \quad (7)$$

Here  $\{\phi_i\}_{i=1}^K$  is an orthonormal basis for  $\mathcal{H}$ ,  $K$  is the dimension of the space  $\mathcal{H}$  (possibly infinite), and  $\lambda_i$  is either  $\sigma$  or  $\beta$ . We denote the set of indices  $i$  with  $\lambda_i = \sigma$  by  $\mathcal{I}_\sigma$  and the set of indices  $i$  with  $\lambda_i = \beta$  by  $\mathcal{I}_\beta$ . We require that  $\mathcal{I}_\sigma \neq \emptyset$ , that  $\mathcal{I}_\beta \neq \emptyset$ , and we get that  $\mathcal{I}_\sigma \cap \mathcal{I}_\beta = \emptyset$  and  $\mathcal{I}_\sigma \cup \mathcal{I}_\beta = \{1, \dots, K\}$ .

First, we show that  $f$  in (5) is defined (finite) for all  $x \in \mathcal{H}$ , even if  $\mathcal{H}$  is infinite dimensional. Obviously  $f(x) \geq 0$  for

all  $x \in \mathcal{H}$ . We also have for arbitrary  $x \in \mathcal{H}$  that

$$f(x) = \sum_{i=1}^K \frac{\lambda_i}{2} \langle x, \phi_i \rangle^2 \leq \frac{\beta}{2} \sum_{i=1}^K \langle x, \phi_i \rangle^2 = \frac{\beta}{2} \|x\|^2 < \infty$$

where the last equality follows from Parseval's identity. Therefore, the optimization problem (3) with  $f$ ,  $g$ , and  $\mathcal{A}$  as in (5), (6), and (7) respectively is well defined also on infinite dimensional spaces.

Next, we show that  $f \in \Gamma_0(\mathcal{H})$  satisfies Assumption 1(i), i.e., that  $f$  is  $\beta$ -smooth and  $\sigma$ -strongly convex.

**Proposition 6:** The function  $f$ , as defined in (5) with  $\lambda_i = \sigma$  for  $i \in \mathcal{I}_\sigma$  and  $\lambda_i = \beta$  for  $i \in \mathcal{I}_\beta$ , is  $\sigma$ -strongly convex and  $\beta$ -smooth.

*Proof.* Since  $\mathcal{H}$  has an orthonormal basis, each element  $x \in \mathcal{H}$  may be decomposed as  $x = \sum_{i=1}^K \langle x, \phi_i \rangle \phi_i$ . We let  $a_i = \langle x, \phi_i \rangle$  and  $b_i = \langle y, \phi_i \rangle$ , to get arbitrary  $x = \sum_{i=1}^K a_i \phi_i \in \mathcal{H}$  and  $y = \sum_{i=1}^K b_i \phi_i \in \mathcal{H}$ . Then

$$\begin{aligned} \frac{\beta}{2} \|x - y\|^2 &= \frac{\beta}{2} \left\| \sum_{i=1}^K a_i \phi_i - \sum_{i=1}^K b_i \phi_i \right\|^2 \\ &= \sum_{i=1}^K \frac{\beta}{2} (a_i - b_i)^2 \geq \sum_{i=1}^K \frac{\lambda_i}{2} (a_i - b_i)^2 \\ &= \sum_{i=1}^K \lambda_i \left( \frac{1}{2} a_i^2 - \frac{1}{2} b_i^2 - \langle b_i \phi_i, a_i \phi_i - b_i \phi_i \rangle \right) \\ &= f(x) - f(y) - \sum_{i=1}^K \langle \lambda_i \langle y, \phi_i \rangle \phi_i, (a_i - b_i) \phi_i \rangle \\ &= f(x) - f(y) - \left\langle \sum_{i=1}^K \lambda_i \langle y, \phi_i \rangle \phi_i, \sum_{i=1}^K (a_i - b_i) \phi_i \right\rangle \\ &= f(x) - f(y) - \langle \nabla f(y), x - y \rangle \end{aligned}$$

where the second equality follows from Riesz-Fischer, the first inequality holds since  $\beta \geq \lambda_i$  for all  $i = 1, \dots, K$ , the third equality follows by expanding the square and noting that  $a_i b_i = \langle a_i \phi_i, b_i \phi_i \rangle$  and  $b_i^2 = \langle b_i \phi_i, b_i \phi_i \rangle$ , the fourth equality follows by identifying the definition of  $f$  in (5) and using  $a_i = \langle x, \phi_i \rangle$  and  $b_i = \langle y, \phi_i \rangle$ , the fifth equality holds since the added cross-terms vanish in the inner product expression due to orthogonality of basis vectors  $\phi_i$ , and the final equality holds by identifying  $x = \sum_{i=1}^K a_i \phi_i$ ,  $y = \sum_{i=1}^K b_i \phi_i$ , and the gradient of  $f$  in (5):

$$\nabla f(x) = \sum_i \lambda_i \langle x, \phi_i \rangle \phi_i.$$

This is the definition of  $\beta$ -smoothness in Definition 3.

An equivalent derivation using  $\sigma$  instead of  $\beta$  and a reversed inequality, shows that  $f$  is also  $\sigma$ -strongly convex.  $\square$

To show that the provided example converges exactly with the rate given in Theorem 1, we need expressions for the proximal operators and reflected proximal operators of  $f$  and  $g$  in (5) and (6) respectively.

*Proposition 7:* The proximal operator of  $f$  in (5) is

$$\text{prox}_{\gamma f}(y) = \sum_{i=1}^K \frac{1}{1+\gamma\lambda_i} \langle y, \phi_i \rangle \phi_i \quad (8)$$

and the reflected proximal operator is

$$R_{\gamma f}(y) = \sum_{i=1}^K \frac{1-\gamma\lambda_i}{1+\gamma\lambda_i} \langle y, \phi_i \rangle \phi_i. \quad (9)$$

*Proof.* We decompose  $x = \sum_{i=1}^K a_i \phi_i$  where  $a_i = \langle x, \phi_i \rangle$  and  $y = \sum_{i=1}^K b_i \phi_i$  where  $b_i = \langle y, \phi_i \rangle$ . Then, for general  $\gamma > 0$ , the proximal operator of  $f$  is given by:

$$\begin{aligned} \text{prox}_{\gamma f}(y) &= \arg \min_x \left\{ \gamma \left( \sum_{i=1}^K \frac{\lambda_i}{2} \langle \phi_i, x \rangle^2 \right) + \frac{1}{2} \|x - y\|^2 \right\} \\ &= \arg \min_{x = \sum_{i=1}^K a_i \phi_i} \left\{ \left( \sum_{i=1}^K \frac{\gamma \lambda_i}{2} a_i^2 \right) + \frac{1}{2} \left\| \sum_{i=1}^K (a_i - b_i) \phi_i \right\|^2 \right\} \\ &= \arg \min_{x = \sum_{i=1}^K a_i \phi_i} \left\{ \frac{1}{2} \sum_{i=1}^K \left( \gamma \lambda_i a_i^2 + (a_i - b_i)^2 \right) \right\} \\ &= \sum_{i=1}^K \arg \min_{a_i} \frac{1}{2} \{ \gamma \lambda_i a_i^2 + (a_i - b_i)^2 \} \phi_i \\ &= \sum_{i=1}^K \frac{1}{1+\gamma\lambda_i} b_i \phi_i = \sum_{i=1}^K \frac{1}{1+\gamma\lambda_i} \langle y, \phi_i \rangle \phi_i. \end{aligned}$$

The reflected resolvent for general  $\gamma > 0$  is given by:

$$\begin{aligned} R_{\gamma f}(y) &= 2\text{prox}_{\gamma f}(y) - y \\ &= 2 \sum_{i=1}^K \frac{1}{1+\gamma\lambda_i} b_i \phi_i - \sum_{i=1}^K b_i \phi_i \\ &= \sum_{i=1}^K \frac{1-\gamma\lambda_i}{1+\gamma\lambda_i} b_i \phi_i = \sum_{i=1}^K \frac{1-\gamma\lambda_i}{1+\gamma\lambda_i} \langle y, \phi_i \rangle \phi_i. \end{aligned}$$

□

The proximal and reflected proximal operators of  $g \equiv 0$  are trivially given by  $\text{prox}_{\gamma g} = R_{\gamma g} = \text{Id}$ .

Next, these results are used to show a lower bound on the convergence rate of Douglas-Rachford splitting for several choices of algorithm parameters  $\alpha$  and  $\gamma$ . First, we state two help lemmas.

*Lemma 1:* The function  $\psi(x) = \frac{1-x}{1+x}$  is a decreasing function for  $x > -1$ .

*Proof.* We have

$$\begin{aligned} &(1-x)/(1+x) < (1-y)/(1+y) \\ \Leftrightarrow &(1-x)(1+y) < (1-y)(1+x) \\ \Leftrightarrow &2y < 2x. \end{aligned}$$

□

*Lemma 2:* For  $x > -1$ , the function  $\psi(x) = \frac{1-x}{1+x}$  satisfies  $\phi(x) \leq -\phi(y)$  if and only if  $y \geq 1/x$ .

*Proof.* We have

$$\begin{aligned} \phi(x) &= (1-x)/(1+x) \leq (y-1)/(1+y) = -\phi(y) \\ \Leftrightarrow &(1-x)(1+y) \leq (y-1)(1+x) \\ \Leftrightarrow &2 \leq 2xy. \end{aligned}$$

□

*Theorem 2:* The generalized Douglas-Rachford splitting algorithm (Algorithm 1) when applied to solve (3) with  $f$ ,  $g$ , and  $\mathcal{A}$  in (5)-(7) converges exactly with rate

$$|1 - \alpha| + \alpha \max \left( \frac{1-\gamma\sigma}{1+\gamma\sigma}, \frac{\gamma\beta-1}{1+\gamma\beta} \right) \quad (10)$$

in the following cases: (i)  $\alpha = 1$  and  $\gamma \in (0, \infty)$ , (ii)  $\alpha \in (0, 1]$  and  $\gamma \in (0, \frac{1}{\sqrt{\sigma\beta}}]$ , (iii)  $\alpha \in [1, \frac{2}{1+\max(\frac{1-\gamma\sigma}{1+\gamma\sigma}, \frac{\gamma\beta-1}{1+\gamma\beta})})$  and  $\gamma \in [\frac{1}{\sqrt{\sigma\beta}}, \infty)$ , (iv)  $\alpha \in (0, \frac{2}{1+\max(\frac{1-\gamma\sigma}{1+\gamma\sigma}, \frac{\gamma\beta-1}{1+\gamma\beta})})$  and  $\gamma = \frac{1}{\sqrt{\sigma\beta}}$ .

*Proof.* For algorithm initial condition  $z^0 = \phi_i$  the Douglas-Rachford algorithm evolves according to

$$z^k = \left( 1 - \alpha + \alpha \frac{1-\gamma\lambda_i}{1+\gamma\lambda_i} \right)^k \phi_i$$

where  $\lambda_i$  is either  $\sigma$  or  $\beta$  depending on if  $i \in \mathcal{I}_\sigma$  or  $i \in \mathcal{I}_\beta$ . This follows immediately from Algorithm 1, the expression of  $R_{\gamma f}$  in Proposition 7, and since  $R_{\gamma g} = \text{Id}$ . Obviously, this converges with rate factor

$$\left| 1 - \alpha + \alpha \frac{1-\gamma\lambda_i}{1+\gamma\lambda_i} \right|.$$

Below, we show for each of the four cases that this rate coincides with the rate (10).

*Case (i):*  $\alpha = 1$  and  $\gamma \in (0, \infty)$

The rate in this case when  $z^0 = \phi_i$ ,  $i \in \mathcal{I}_\sigma$ , is exactly  $\left| \frac{1-\gamma\sigma}{1+\gamma\sigma} \right|$ . The rate when  $z^0 = \phi_i$ ,  $i \in \mathcal{I}_\beta$ , is exactly  $\left| \frac{1-\gamma\beta}{1+\gamma\beta} \right|$ . A lower bound on the convergence of the algorithm when  $\alpha = 1$  is therefore

$$\begin{aligned} \max \left( \left| \frac{1-\gamma\sigma}{1+\gamma\sigma} \right|, \left| \frac{1-\gamma\beta}{1+\gamma\beta} \right| \right) &= \max \left( \frac{1-\gamma\sigma}{1+\gamma\sigma}, \frac{\gamma\beta-1}{1+\gamma\beta} \right) \\ &= |1 - \alpha| + \alpha \max \left( \frac{1-\gamma\sigma}{1+\gamma\sigma}, \frac{\gamma\beta-1}{1+\gamma\beta} \right). \end{aligned}$$

where the first equality is due to Lemma 1, and the second holds since  $\alpha = 1$ . This proves the first claim.

*Case (ii):*  $\alpha \in (0, 1]$  and  $\gamma \in (0, \frac{1}{\sqrt{\sigma\beta}}]$

The rate when using initial condition  $z^0 = \phi_i$ ,  $i \in \mathcal{I}_\sigma$ , is  $r_\sigma := 1 - \alpha + \alpha \frac{1-\gamma\sigma}{1+\gamma\sigma}$  (since  $(1-\alpha) \geq 0$  and  $\alpha \frac{1-\gamma\sigma}{1+\gamma\sigma} \geq 0$ ). For  $z^0 = \phi_i$ ,  $i \in \mathcal{I}_\beta$ , and  $\gamma \leq \frac{1}{\sqrt{\sigma\beta}}$ , we get

$$\begin{aligned} |1 - \alpha + \alpha \frac{1-\gamma\beta}{1+\gamma\beta}| &\leq |1 - \alpha| + \alpha \frac{1-\gamma\beta}{1+\gamma\beta} \\ &= 1 - \alpha + \alpha \frac{1-\gamma\beta}{1+\gamma\beta} \\ &\leq 1 - \alpha + \alpha \frac{1-\gamma\sigma}{1+\gamma\sigma} = r_\sigma \end{aligned}$$

where the last inequality holds due to Lemma 1. For  $z^0 = \phi_i$ ,  $i \in \mathcal{I}_\beta$ , and  $\gamma \in [\frac{1}{\beta}, \frac{1}{\sqrt{\sigma\beta}}]$ , we get

$$\begin{aligned} |1 - \alpha + \alpha \frac{1-\gamma\beta}{1+\gamma\beta}| &\leq |1 - \alpha| + |\alpha \frac{1-\gamma\beta}{1+\gamma\beta}| \\ &= 1 - \alpha + \alpha \frac{\gamma\beta-1}{1+\gamma\beta} \\ &\leq 1 - \alpha + \alpha \frac{1-\gamma\sigma}{1+\gamma\sigma} = r_\sigma \end{aligned}$$

where the last inequality follows from Lemma 2. Thus, a lower bound on the rate for  $\alpha \in (0, 1]$  and  $\gamma \in (0, \frac{1}{\sqrt{\sigma\beta}}]$  is

$$r_\sigma = 1 - \alpha + \alpha \frac{1-\gamma\sigma}{1+\gamma\sigma} = |1 - \alpha| + \alpha \max\left(\frac{1-\gamma\sigma}{1+\gamma\sigma}, \frac{\gamma\beta-1}{1+\gamma\beta}\right).$$

This proves the second claim.

*Case (iii):*  $\alpha \in [1, \frac{2}{1+\max(\frac{1-\gamma\sigma}{1+\gamma\sigma}, \frac{\gamma\beta-1}{1-\gamma\beta})}]$  and  $\gamma \in [\frac{1}{\sqrt{\sigma\beta}}, \infty)$

The rate when using  $z^0 = \phi_i$ ,  $i \in \mathcal{I}_\beta$ , is  $r_\beta := \alpha - 1 + \alpha \frac{\gamma\beta-1}{1+\gamma\beta}$  (since  $(1 - \alpha) \leq 0$  and  $\alpha \frac{1-\gamma\beta}{1+\gamma\beta} \leq 0$ ). For  $z^0 = \phi_i$ ,  $i \in \mathcal{I}_\sigma$ , and  $\gamma \in [\frac{1}{\sqrt{\sigma\beta}}, \frac{1}{\sigma}]$  the rate is

$$\begin{aligned} |1 - \alpha + \alpha \frac{1-\gamma\sigma}{1+\gamma\sigma}| &\leq |1 - \alpha| + |\alpha \frac{1-\gamma\sigma}{1+\gamma\sigma}| \\ &= \alpha - 1 + \alpha \frac{1-\gamma\sigma}{1+\gamma\sigma} \\ &\leq \alpha - 1 + \alpha \frac{\gamma\beta-1}{1+\gamma\beta} = r_\beta \end{aligned}$$

where the last inequality follows from Lemma 2. For  $z^0 = \phi_i$ ,  $i \in \mathcal{I}_\sigma$ , and  $\gamma \geq \frac{1}{\sigma}$ , we get

$$\begin{aligned} |1 - \alpha + \alpha \frac{1-\gamma\sigma}{1+\gamma\sigma}| &\leq |1 - \alpha| + |\alpha \frac{1-\gamma\sigma}{1+\gamma\sigma}| \\ &= \alpha - 1 + \alpha \frac{\gamma\sigma-1}{1+\gamma\sigma} \\ &\leq \alpha - 1 + \alpha \frac{\gamma\beta-1}{1+\gamma\beta} = r_\beta \end{aligned}$$

where the last inequality is due to Lemma 1. This implies that a lower bound on the rate for  $\alpha \in [1, \frac{2}{1+\max(\frac{1-\gamma\sigma}{1+\gamma\sigma}, \frac{\gamma\beta-1}{1-\gamma\beta})}]$

and  $\gamma \in [\frac{1}{\sqrt{\sigma\beta}}, \infty)$  is

$$r_\beta = \alpha - 1 + \alpha \frac{\gamma\beta-1}{1+\gamma\beta} = |1 - \alpha| + \alpha \max\left(\frac{1-\gamma\sigma}{1+\gamma\sigma}, \frac{\gamma\beta-1}{1+\gamma\beta}\right).$$

*Case (iv):*  $\alpha \in (0, \frac{2}{1+\max(\frac{1-\gamma\sigma}{1+\gamma\sigma}, \frac{\gamma\beta-1}{1-\gamma\beta})})$  and  $\gamma = \frac{1}{\sqrt{\sigma\beta}}$

This Case follows directly from Cases (ii) and (iii).  $\square$

The convergence rate for the example given by  $f$  and  $g$  in (5) and (6) respectively coincides with the upper bound on the convergence rate in [10, Theorem 1] (which is restated in Theorem 1). The bound in [10, Theorem 1] is therefore tight for the class of problems under consideration and for the combination of algorithm parameters specified in Theorem 2.

*Remark 5:* The upper bound on the rate in [10, Theorem 1], relies on the triangle inequality between  $(1 - \alpha)(z^k - \bar{z})$  and  $\alpha(R_{\gamma g} R_{\gamma f} z^k - R_{\gamma g} R_{\gamma f} \bar{z})$ . To get equality, we must find  $\alpha$ ,  $\gamma$  and  $z^k$  such that  $(1 - \alpha)(z^k - \bar{z})$  and  $\alpha(R_{\gamma g} R_{\gamma f} z^k - R_{\gamma g} R_{\gamma f} \bar{z})$  are parallel. For remaining combinations of  $\gamma$  and  $\alpha$ , these become anti-parallel, and the rate bound is not met exactly. Note, however, that for optimal choices of  $\alpha$  and  $\gamma$ , the bound is tight.

## B. Dual Douglas-Rachford splitting (ADMM)

This section concerns tightness of the rate bounds when Douglas-Rachford splitting is applied to the dual problem (4), or equivalently, when ADMM applied to the primal problem (3). To show tightness in this case, we consider the following problem

$$f(x) = \sum_{i=1}^K \frac{\lambda_i}{2} \langle x, \phi_i \rangle^2 \quad (11)$$

$$g(x) = \iota_{x=0}(x) \quad (12)$$

$$\mathcal{A}(x) = \sum_{i=1}^K \nu_i \langle x, \phi_i \rangle \phi_i \quad (13)$$

where  $\lambda_i = \sigma$  and  $\nu_i = \theta > 0$  if  $i \in \mathcal{I}_\sigma$  and  $\lambda_i = \beta$  and  $\nu_i = \zeta > \theta$  if  $i \in \mathcal{I}_\beta$ , where  $\mathcal{I}_\sigma$  and  $\mathcal{I}_\beta$  are the same as before. That  $\mathcal{A}$  is linear follows trivially. That it is self-adjoint, bounded, and surjective is shown in the following proposition.

*Proposition 8:* The linear operator  $\mathcal{A}$  defined in (13) is self-adjoint, i.e.  $\mathcal{A} = \mathcal{A}^*$ , and for every  $x \in \mathcal{H}$ , we have

$$\theta \|x\| \leq \|\mathcal{A}(x)\| \leq \zeta \|x\|. \quad (14)$$

Further  $\|\mathcal{A}\| = \|\mathcal{A}^*\| = \zeta$ .

*Proof.* We start by showing that  $\mathcal{A}$  is self-adjoint. We have

$$\begin{aligned} \langle \mathcal{A}(x), \mu \rangle &= \left\langle \sum_{i=1}^K \nu_i \langle x, \phi_i \rangle \phi_i, \sum_{i=1}^K \langle \mu, \phi_i \rangle \phi_i \right\rangle \\ &= \sum_{i=1}^K \langle \nu_i \langle x, \phi_i \rangle \phi_i, \langle \mu, \phi_i \rangle \phi_i \rangle \\ &= \sum_{i=1}^K \langle \langle x, \phi_i \rangle \phi_i, \nu_i \langle \mu, \phi_i \rangle \phi_i \rangle \\ &= \left\langle \sum_{i=1}^K \langle x, \phi_i \rangle \phi_i, \sum_{i=1}^K \nu_i \langle \mu, \phi_i \rangle \phi_i \right\rangle \\ &= \langle x, \mathcal{A}(\mu) \rangle \end{aligned}$$

where moving of summations are due to orthogonality of  $\phi_i$ . Next we show the first inequality in (14):

$$\begin{aligned} \|\mathcal{A}(x)\| &= \left\| \theta \sum_{i \in \mathcal{I}_\sigma} \langle x, \phi_i \rangle \phi_i + \zeta \sum_{i \in \mathcal{I}_\beta} \langle x, \phi_i \rangle \phi_i \right\| \\ &\geq \theta \left\| \sum_{i=1}^K \langle x, \phi_i \rangle \phi_i \right\| = \theta \|x\| \end{aligned}$$

since  $0 < \theta \leq \zeta$ . The second inequality in (14) is proven similarly. Finally, we show  $\|\mathcal{A}\| = \zeta$ . We have already shown that  $\|\mathcal{A}(x)\| \leq \zeta \|x\|$  for all  $x \in \mathcal{H}$ , i.e., that  $\|\mathcal{A}\| \leq \zeta$ . By definition of the operator norm, we also know that  $\|\mathcal{A}\| \geq \|\mathcal{A}(x)\|$  for all  $x \in \mathcal{H}$  with  $\|x\| \leq 1$ . Choosing  $x = \phi_j$  (which satisfies  $\|x\| = \|\phi_j\| = 1$ ) for any  $j \in \mathcal{I}_\beta$  (i.e.  $j$  with  $\nu_j = \zeta$ ) gives

$$\|\mathcal{A}\| \geq \|\mathcal{A}(\phi_j)\| = \left\| \sum_{i=1}^K \nu_i \langle \phi_j, \phi_i \rangle \phi_i \right\| = \|\nu_j\| = \zeta.$$

Thus,  $\|\mathcal{A}\| = \zeta$  and the proof is complete.  $\square$

This result implies that the assumptions in [10, Corollary 2] (and Proposition 5) are met by  $f$ ,  $g$ , and  $\mathcal{A}$  in (11), (12), and (13) respectively. The bound on the convergence rate from [10, Corollary 2] (and restated in Proposition 5) is therefore valid. To show that this bound is tight for the class of problems under consideration, we need the following explicit characterization of  $d$ :

$$\begin{aligned}
d(\mu) &:= f^*(-\mathcal{A}^*\mu) = f^*(-\mathcal{A}\mu) \\
&= \sup_x \{ \langle -\mathcal{A}\mu, x \rangle - f(x) \} \\
&= -\inf_x \{ f(x) + \langle \mathcal{A}\mu, x \rangle \} \\
&= -\inf_x \left\{ \sum_{i=1}^K \frac{\lambda_i}{2} \langle x, \phi_i \rangle^2 + \left\langle \sum_{i=1}^K \nu_i \langle \mu, \phi_i \rangle \phi_i, x \right\rangle \right\} \\
&= -\inf_{a_i} \left\{ \sum_{i=1}^K \frac{\lambda_i}{2} \left\langle \sum_{i=1}^K a_i \phi_i, \phi_i \right\rangle^2 \right. \\
&\quad \left. + \left\langle \sum_{i=1}^K \nu_i \langle \mu, \phi_i \rangle \phi_i, \sum_{i=1}^K a_i \phi_i \right\rangle \right\} \\
&= -\sum_{i=1}^K \inf_{a_i} \left\{ \frac{\lambda_i}{2} \langle a_i \phi_i, \phi_i \rangle^2 + \langle \nu_i \langle \mu, \phi_i \rangle \phi_i, a_i \phi_i \rangle \right\} \\
&= -\sum_{i=1}^K \inf_{a_i} \left\{ \frac{\lambda_i}{2} a_i^2 + \nu_i \langle \mu, \phi_i \rangle a_i \right\} \\
&= -\sum_{i=1}^K \left\{ \frac{(\nu_i \langle \mu, \phi_i \rangle)^2}{2\lambda_i} - \frac{(\nu_i \langle \mu, \phi_i \rangle)^2}{\lambda_i} \right\} \\
&= \sum_{i=1}^K \frac{(\nu_i \langle \mu, \phi_i \rangle)^2}{2\lambda_i} = \sum_{i=1}^K \frac{\nu_i^2}{2\lambda_i} \langle \mu, \phi_i \rangle^2
\end{aligned}$$

where the decomposition  $x = \sum_{i=1}^K a_i \phi_i$  with  $a_i = \langle x, \phi_i \rangle$  is used, and the optimal  $a_i = -\nu_i \langle \mu, \phi_i \rangle / \lambda_i$ . The function  $d$  has exactly the same structure as the function  $f$  but with  $\lambda_i$  in  $f$  in (5) replaced by  $\nu_i^2 / \lambda_i$  in  $d$ . The function  $g^*$  is, for all  $\mu \in \mathcal{H}$ , given by

$$g^*(\mu) = \sup_{x \in \mathcal{H}} \{ \langle \mu, x \rangle - \nu_{x=0}(x) \} = \langle \mu, 0 \rangle = 0.$$

This implies that the dual problem (4) with  $f$ ,  $g$ , and  $\mathcal{A}$  specified in (11), (12), and (13) has exactly the same structure as the primal problem (3) with  $f$  and  $g$  specified in (5) and (6) respectively and with  $\mathcal{A} = \text{Id}$ . The only things that differ are the scalars that multiply the quadratic terms in the functions  $f$  and  $d$  respectively. Therefore, we can immediately state the following corollary to Theorem 2.

*Corollary 1:* Let  $f$  be given by (11),  $g$  be given by (12), and  $\mathcal{A}$  be given by (13). Then the generalized Douglas-Rachford algorithm applied to solve the dual problem (4) (or equivalently ADMM applied to solve (3)) converges as in Theorem 2 with  $\beta$  and  $\sigma$  in Theorem 2 replaced by  $\hat{\beta} = \frac{\|\mathcal{A}\|^2}{\sigma}$  and  $\hat{\sigma} = \frac{\theta^2}{\beta}$  respectively.

The exact rate provided in Corollary 1, coincides with rate bound in [10, Corollary 2] and Proposition 5. Therefore,

we conclude that the rate bound in [10, Corollary 2] for ADMM on the primal problem, or equivalently for Douglas-Rachford splitting on the dual problem, is tight for the class of problems under consideration for many algorithm parameter choices. Especially, the bound is tight for the optimal parameters  $\alpha$  and  $\gamma$ , as in the primal Douglas-Rachford case.

## VI. CONCLUSION

Recent results in the literature have shown linear convergence of Douglas-Rachford splitting and ADMM under various assumptions. In this paper, we have shown that the linear convergence rate bounds presented in [10] are indeed tight for the class of problems under consideration.

## REFERENCES

- [1] D. Davis and W. Yin. Convergence rate analysis of several splitting schemes. Available <http://arxiv.org/abs/1406.4834>, August 2014.
- [2] D. Davis and W. Yin. Faster convergence rates of relaxed Peaceman-Rachford and ADMM under regularity assumptions. Available: <http://arxiv.org/abs/1407.5210>, July 2014.
- [3] L. Demanet and X. Zhang. Eventual linear convergence of the Douglas-Rachford iteration for basis pursuit. Available: <http://arxiv.org/abs/1301.0542>, May 2013.
- [4] W. Deng and W. Yin. On the global and linear convergence of the generalized alternating direction method of multipliers. Technical Report CAAM 12-14, Rice University, 2012.
- [5] J. Douglas and H. H. Rachford. On the numerical solution of heat conduction problems in two and three space variables. *Trans. Amer. Math. Soc.*, 82:421–439, 1956.
- [6] D. Gabay. Applications of the method of multipliers to variational inequalities. In M. Fortin and R. Glowinski, editors, *Augmented Lagrangian Methods: Applications to the Solution of Boundary-Value Problems*. North-Holland: Amsterdam, 1983.
- [7] D. Gabay and B. Mercier. A dual algorithm for the solution of nonlinear variational problems via finite element approximation. *Computers and Mathematics with Applications*, 2(1):17–40, 1976.
- [8] E. Ghadimi, A. Teixeira, I. Shames, and M. Johansson. Optimal parameter selection for the alternating direction method of multipliers (ADMM): Quadratic problems. 2013. Submitted. Available: <http://arxiv.org/abs/1306.2454>.
- [9] P. Giselsson and S. Boyd. Diagonal scaling in Douglas-Rachford splitting and ADMM. In *53rd IEEE Conference on Decision and Control*, Los Angeles, CA, December 2014. Accepted for publication.
- [10] P. Giselsson and S. Boyd. Metric selection in Douglas-Rachford splitting and ADMM. 2014. Submitted.
- [11] R. Glowinski and A. Marroco. Sur l'approximation, par éléments finis d'ordre un, et la résolution, par pénalisation-dualité d'une classe de problèmes de dirichlet non linéaires. *ESAIM: Mathematical Modelling and Numerical Analysis - Modélisation Mathématique et Analyse Numérique*, 9:41–76, 1975.
- [12] B. He and X. Yuan. On the  $o(1/n)$  convergence rate of the Douglas-Rachford alternating direction method. *SIAM Journal on Numerical Analysis*, 50(2):700–709, 2012.
- [13] M. Hong and Z.-Q. Luo. On the linear convergence of the alternating direction method of multipliers. Available: <http://arxiv.org/abs/1208.3922>, March 2013.
- [14] P. L. Lions and B. Mercier. Splitting algorithms for the sum of two nonlinear operators. *SIAM Journal on Numerical Analysis*, 16(6):964–979, 1979.
- [15] R. Nishihara, L. Lessard, B. Recht, A. Packard, and M. Jordan. A general analysis of the convergence of ADMM. February 2015. Available: <http://arxiv.org/abs/1502.02009>.
- [16] P. Patrinos, L. Stella, and A. Bemporad. Douglas-Rachford splitting: Complexity estimates and accelerated variants. In *Proceedings of the 53rd IEEE Conference on Decision and Control*, Los Angeles, CA, December 2014.
- [17] D. W. Peaceman and H. H. Rachford. The numerical solution of parabolic and elliptic differential equations. *Journal of the Society for Industrial and Applied Mathematics*, 3(1):28–41, 1955.

- [18] A. Raghunathan and S. Di Cairano. ADMM for convex quadratic programs: Linear convergence and infeasibility detection. November 2014. Available: <http://arxiv.org/abs/1411.7288>.
- [19] R. T. Rockafellar. *Convex Analysis*, volume 28. Princeton University Press, Princeton, NJ, 1970.
- [20] Michel Willem. *Functional Analysis*. Springer New York, 2013.